Problem for the week of January 18, 2010

Let A be the following $n \times n$ matrix

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-2} & -a_{n-1} \end{bmatrix}$$

Show that the characteristic polynomial of A is $\det(A - tI) = (-1)^n p(t)$, where

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

The matrix A is known as the *companion matrix* of the polynomial $p(t)$.

Solution

設計下面的 $n \times n$ 階下三角形矩陣

$$B = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ t & 1 & 0 & \cdots & 0 \\ t^2 & t & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t^{n-1} & t^{n-2} & t^{n-3} & \cdots & 1 \end{bmatrix}$$

令 $C = (A - tI)B$, 乘開得到

$$C = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -p(t) & * & * & \cdots & * \end{bmatrix}$$

上式中 $*$ 表示某多項式, 對第一行以共因子展開計算行列式

$$\det C = (-1)^{n+1}(-p(t))\det I_{n-1} = (-1)^n p(t)$$

又因爲 B 爲下三角形矩陣且主對角元皆爲 1, 可知 $\det B = 1$, 就有

$$\det C = \det((A - tI)B) = \det(A - tI)(\det B) = \det(A - tI)$$

□